Back reaction of the neutrino field in an Einstein universe

M.B. Altaie[*]

Department of Physics, Yarmouk University, Irbid-Jordan

Abstract

The back reaction effect of the neutrino field at finite temperature in the background of the static Einstein universe is investigated. A relationship between the temperature of the universe and its radius is found. As in the previously studied cases of the massless scalar field and the photon field, this relation exhibit a minimum radius below which no self-consistent solution for the Einstein field equation can be found. A maximum temperature marks the transition from a vacuum dominated state to the radiation dominated state universe. In the light of the results obtained for the scalar, neutrino and photon fields the role of the back reaction of quantum fields in controling the value of the cosmological constant is briefly discussed.

[*]Electronic Adress: maltaie@yu.edu.jo

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I. Introduction

The discovery of the Cosmic Background radiation (CMB) [1] revived the theory of the hot origin of the universe (the big-bang model). The CMB was originally a prediction of the theory of Gamow and his collaborators which was worked out in the late 1940's in the context of investigating the origin of the natural abundance of elements. The most refined analysis along this line predicted a cosmic background radiation at a temperature about 5 K (for a concise recent review of the subject see ref. [2]) . However, since the Gamow model started with the universe at the times when the temperature was about 10^{12} K, the new interest in the origin of the universe sought much earlier times at much higher temperatures. In the light of the absence of a full quantum theory of gravity, the works dealing with the state of the universe at very early times had to consider the quantized matter fields in the classical background of the universe as described by the theory of general relativity. Matter fields were brought into connection with spacetime curvature through the calculation of the vacuum expectation value of the energy-momentum tensor $< 0|T_{\mu\nu}|0 >$ [3-7]. In these works and the similar ones that followed, quantum gravity and matter are truncated at the one loop level. For free matter fields, there are no higher loop processes anyway. The contribution from graviton is of zeroth order in G. As the loop expansion is an expansion in $\hbar$, the theory truncated at the one loop level contains all terms of the complete theory of order $\hbar$ and is, in that sense, the first order quantum correction to general relativity [8].

The motivations for studying the vacuum expectation value of the energy-momentum tensor stems from the fact that $T_{\mu\nu}$ is a local quantity that can be defined at a specific spacetime point, contrary to the particle concept which is global. The energy-momentum tensor also acts as a source of gravity in the Einstein field equations, therefore $< 0|T_{\mu\nu}|0 >$ plays an important role in any attempt to model a self-consistent dynamics involving the classical gravitational field coupled to the quantized matter fields. So, once $< 0|T_{\mu\nu}|0 >$ is calculated in a specified background geometry, we can substitute it on the RHS of the Einstein field equations and demand self consistency, i.e.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = -8\pi < 0|T_{\mu\nu}|0 >, \qquad (1)$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature, and Λ is the so-called cosmological constant.

The solution of (1) will determine the development of the spacetime in presence of the given matter field, for which the vacuum state $|0 >$ can be unambiguously defined. This is known as the "back reaction problem". It is interesting to perform the calculation of $< 0|T_{\mu\nu}|0 >$ in Friedman-Robertson-Walker (FRW) models because the real universe is, more or less, a sophisticated form of the Friedman models. However the time-dependence of the spacetime metric generally creates unsolvable fundamental problems. One such a problem was the definition of vacuum in a time-dependent background [9]; a time-dependent

background is eligible for producing particles continuously, therefore, pure vacuum states in the Minkowskian sense do not exist. Also an investigation into the thermodynamics of a time-dependent systems lacks the proper definition of thermal equilibrium, which is a basic necessity for studying finite-temperature field theory in curved backgrounds.[10].

The static Einstein universe stands the above two fundamental challenges. First, being static, the Einstein universe leaves no ambiguity in defining the vacuum both locally and globally. The same feature also allows for thermal equilibrium to be defined unambiguously. Furthermore, the Einstein static metric is conformal to all Robertson-Walker metrics, and this property enabled Kennedy [10] to show that the thermal Green functions for the static Einstein universe and the time-dependent Robertson-Walker universe are conformally related, hence deducing a (1-1) correspondence between the vacuum and the many particle states of both universes. So that, under the equilibrium condition, the thermodynamics of quantum fields in an Einstein universe of radius a is equivalent to that of an instantaneously static Friedman-Roberson-Walker universe of equal radius [3,6, 11]. This means that the results obtained in FRW universe would be qualitatively the same as those obtained in an Einstein universe.

Although not a realistic cosmological model, the Einstein universe remains to be a useful theoretical tool to achieve better understanding of the interplay of spacetime curvature and of quantum field theoretic effects. The recent findings of Plunien *et al.* [12] that finite temperatures can enhance the pure vacuum effect by several orders of magnitude, can be used to explain the behavior of the system during the Casimir (vacuum) regime. Since this means that the finite temperature corrections will surely enhance the positive vacuum energy density of closed system causing it to behave, thermodynamically, as being controlled by the vacuum energy.

Recently [13] (hereafter will be referred to as I) we have investigated the back reaction effects of the conformally coupled massless scalar field and the photon field at finite temperatures in the background of the Einstein static universe. In each case we found a relationship between the temperature of the system and the radius of the Einstein universe. This relation exhibit a minimum radius below which no self-consistent solution for the Einstein field equations can be found. A maximum temperature is also spotted marking the transition from the vacuum dominated state to the radiation dominated state.

Motivated by the above findings, and in order to complete the picture of the back reaction of the spinor fields at finite temperatures in the Einstein universe, we will investigate in this paper the case of the neutrino field and will deduce the relationship between the temperature and the radius, and will consequently find the minimum radius and the maximum temperature. Upon the availability of all results we will compare the contributions of all the three fields, the scalar, neutrino and the photon fields. Some interesting results will be discussed.

We will also consider the variation of the cosmological constant with the temperature of the Einstein universe in presence of the neutrino field, and then will compile the results for the massless scalar field, the neutrino field, and

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the photon field for comparison. The analysis shows that the smallness of the present value of the cosmological constant is understandable in the context of the behavior of massless quantum fields in this universe if they will play the role of energy -momentum source. Also, we notice that the contribution from the scalar field is higher, by an order of magnitude, than the contributions of the other fields. Throughout this paper we will use the natural units in which $G = c = k = \hbar = 1$.

II.   Basic Formalism

The Einstein static universe is one solution of the modified Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = -8\pi T_{\mu\nu} \tag{2}$$

The metric of the Einstein static universe is given by

$$ds^2 = dt^2 - a^2 \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] , \tag{3}$$

where a is the radius of the spatial part of the universe S^3 and, $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$.

We consider an Einstein static universe being filled with massless neutrino gas in thermal equilibrium at temperature T. The total energy density of the system can be written as

$$< T_{00} >_{tot} = < T_{00} >_T + < T_{00} >_0, \tag{4}$$

where $< T_{00} >_0$ is the zero-temperature vacuum energy density and $< T_{00} >_T$ is the corrections for finite temperatures, which can be calculated using the mode sum

$$< T_{00} >_T = \frac{1}{V} \sum_n \frac{d_n \epsilon_n}{\exp \beta \epsilon_n + 1}, \tag{5}$$

where ϵ_n and d_n are the eigen energies and degeneracies of the nth state, and $V = 2\pi^2 a^3$ is the volume of the spatial section of the Einstein universe.

The back reaction of the field on the background spacetime can be studied by substituting for $< T_{\mu\nu} >_{tot}$ on the RHS of the Einstein field equation (1), i.e.,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = -8\pi < T_{\mu\nu} >_{tot} . \tag{6}$$

All the Einstein field equations for the system are satisfied due to the symmetry of the Einstein universe which is topologically described by $T \otimes S^3$, and also due to the structure of $< T_{\mu\nu} >$ in this geometry which is diagonal and traceless, and is given by (see [8],p.186)

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$$< T_\mu^\nu >=< T_0^0 > diag(1, -1/3, -1/3, -1/3), \tag{7}$$

Since we are interested in the energy density, we will consider T_{00} only. In order to eliminate Λ from (6) we multiply both sides with $g_{\mu\nu}$ and sum over μ and ν, then using the fact that $T_\mu^\mu = 0$ for massless fields, and for the Einstein universe $R_{00} = 0$, $g_{00} = 1$,and $R = \frac{6}{a^2}$, we get

$$\frac{6}{a^2} = 32\pi < T_{00} >_{tot} . \tag{8}$$

Note that in the general case conformal anomalies do appear in the expression for $< T_\mu^\mu >$, but because of the high symmetry enjoyed by the Einstein universe these anomalies do not appear and $< T_\mu^\mu >$ is found to be traceless for massless particles.

III. THE VACUUM ENERGY AND BACK REACTION

The solution of the Dirac equation in an Einstein universe has been considered by Schrodinger [14] and more recently by Unruh [15]. The eigen energies are found to be

$$\epsilon_n = \frac{1}{a}(n + \frac{1}{2}), \tag{9}$$

with degeneracy, for the four-component neutrino

$$d_n = 4n(n+1), \tag{10}$$

The zero-temperature vacuum energy density of the neutrino field in the background of an Einstein universe is given by [3]

$$< T_{00} >_0 = \frac{17}{960\pi^2 a^4}. \tag{11}$$

Neutrinos obey Fermi-Dirac statistics. Thus the total energy density of the system at finite temperature can be written, in terms of the mode-sum in Eq. (5), as

$$< T_{00} >_{tot} = \frac{1}{\pi^2 a^4} \sum_{n=1}^{\infty} \frac{n(n+1/2)(n+1)}{\exp(n+1/2)/Ta + 1} + \frac{17}{960\pi^2 a^4} \tag{12}$$

In the low-temperature limit the result reduce to [15]

$$\lim_{\xi \longrightarrow 0} < T_{00} >_{tot} = \frac{17}{960\pi^2 a^4}. \tag{13}$$

Substituting this in Eq. (8) we get

$$a_{0\nu} = \left(\frac{17}{180\pi} \right)^{1/2} l_p. \tag{14}$$

This is the minimum radius for an Einstein static universe filled with massless neutrinos at finite temperatures. Note that $a_{0\nu}$ here is less than one Planckian length l_p, this goes beyond the range of validity of the quasi-classical approximation adopted in the present work. But fortunately, the region of validity of the approach can be extended if one takes the number of fields large enough (see for instance Ref. [17]).

In the high temperature (or large radius) limit the result is [16]

$$\lim_{\xi \longrightarrow \infty} < T_{00} >_{tot} = \frac{7}{60} \pi^2 T^4 - \frac{T^2}{24a^2}, \tag{15}$$

where $\xi = Ta$.

The back-reaction of the field can be studied if we substitute Eq. (12) in Eq. (8), where this time we obtain

$$a^2 = \frac{16}{3\pi} \sum_{n=1}^{\infty} \frac{n(n+1/2)(n+1)}{\exp(n+1/2)/Ta+1} + \frac{17}{180\pi}. \tag{16}$$

The solutions for this equation are depicted in Fig. 1, where we see that the behavior is qualitatively the same as that encountered in the conformally coupled scalar field case and the photon case in I. Again, two regimes are recognized: one corresponding to small values of ξ where the temperature rises sharply reaching a maximum at $T_{max} \approx 1.076 T_p = 1.52 \times 10^{32}$ K at a radius $a_t \approx 0.204 l_p = 3.2 \times 10^{-34}$ cm. Since this regime is controlled by the vacuum energy (the Casimir energy), therefore we prefer to call it the "Casimir regime". The second regime is what we call the "Planck regime", which correspond to large values of ξ, and in which the temperature asymptotically approaches zero for very large values of a.

The background (Tolman) temperature of the neutrino field is

$$T_{b\nu} = \left(\frac{45}{28\pi^3 a^2} \right)^{1/4} \tag{17}$$

At a radius of 1.38×10^{28} cm (the present Hubble length), we obtain a background temperature of 23.06 K, and if we require the background temperature to have the same value as the average value of the neutrino background temperature of 1.94 K, the radius of the Einstein universe has to be 1.94×10^{30} cm. Again more than two orders of magnitude larger than the estimated value of Hubble length.

Comparing the temperature for neutrinos from Eq. (17) with Eq. (21) from I, we immediately see that the ratio of the neutrino temperature to the photon temperature is

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$$\frac{T_{b\nu}}{T_{b\gamma}} = \left(\frac{4}{7}\right)^{1/4} = 0.869 \tag{18}$$

This may be compared with the standard ratio of the neutrino temperature to the photon temperature, generated in excess because of the heating of the photons by the electron positron annihilation, calculated according to the standard big bang scenario (see [18], p.537)

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} = 0.714 \tag{19}$$

Fig 2. compiles the temperature-radius relationships for the scalar, neutrino and photon fields, and table (1) summarizes the results of the present work and the previous ones reported in I. Note that we quote here the correct value for the Tolman temperature of the photon field $T_{b\gamma}$ which was mistakenly given in I as 30.267 K.

Table (1) Comparison between the parameters for the massless fields

Field	a_0	$a(T_{\max})$	$T_{\max}$	T_b K
Scalar	0.059	0.072	2.218	31.55
Neutrino	0.173	0.204	1.076	23.06
Photon	0.279	0.340	1.015	26.53

IV. The Cosmological Constant

The cosmological constant was introduced by Einstein in order to account for Mach principle and to justify the equilibrium of a static universe against its own gravitational attraction. The possibility that the universe may be expanding led Einstein to abandon the idea of a static universe and, along with it the cosmological constant. However the Einstein static universe remained to be of interest to theoreticians since it provided a useful model to achieve better understanding of the interplay of spacetime curvature and of quantum field theoretic effects. Recent years have witnessed a resurgence of interest in the possibility that a positive cosmological constant Λ may dominate the total energy density in the universe (for recent reviews see [19] and [20]). This interest stems from the recent observations of high redshift type Ia supernovae, which appear to suggest that the universe is accelerating with large fraction of the cosmological density in the form of a cosmological constant Λ. At a theoretical level Λ is predicted to arise out of the zero-point quantum vacuum fluctuations of the fundamental quantum fields. Using parameters arising in the electroweak theory results in a value of the vacuum energy density $\rho_{vac} = 10^6$ GeV^4 which is almost 10^{53} times larger than the current observational upper limit on Λ which is 10^{-47} GeV$^4 \sim 10^{-29}$ gm/cm^3. On the other hand the QCD vacuum is expected to generate a cosmological constant of the order of 10^{-3} GeV^4 which is many orders of magnitude larger than the observed value. This is known as the

old cosmological constant problem. The new cosmological constant problem is to understand why ρ_{vac} is not only small but also, as the current observations seem to indicate, is of the same order of magnitude as the present mass density of the universe.

In what follows we are going to investigate how the value of the cosmological constant changes in the context of the presence of the massless quantum fields in the background of an Einstein universes having different temperatures (or radii) as a consequence of the back reaction effect. This investigation may be considered "phenomenological", in the sense that no attempt is made to derive the model from an underlaying quantum field theory in contrast to the customary approach normally adopted for such calculations.

Contracting the field equations in (6) we find that

$$\Lambda = \frac{R}{4} = \frac{3}{2a^2}.$$ (20)

On the other hand the Einstein field equations reduces to

$$-\frac{3}{a^2} + \Lambda = -8\pi\rho_{tot},$$ (21)

and

$$-\frac{1}{a^2} + \Lambda = \frac{8\pi\rho_{tot}}{3}$$ (22)

where $\rho_{tot} = <T_0^0>_{tot}$. Solving the above two equations we obtain

$$\Lambda = 8\pi\rho_{tot}$$ (23)

Using (20) and the results obtained in the previous section for the dependence of T on a we can solve for the dependence of Λ on T. Fig. (3) depicts the relationship between the cosmological constant Λ and the temperature for successive states of the Einstein universe under the effect of back reaction of the neutrino field at finite temperatures. It shows that the back reaction of finite-temperature quantum fields in this model provide a large value for the cosmological constant during the Casimir regime (vacuum dominated state of the universe). From the point of view of inflationary models this large value of Λ is needed to resolve the problem of horizon and the problem of flatness, and possibly to generate seed fluctuations for galaxy formation [19]. On the other hand the value of the cosmological constant today, according to the a above analysis, should be very small.

Fig. (4) compiles the data for the three massless fields: the scalar, the neutrino and the photon field using the respective results for the radius temperature relationship reported in I. It is clear that the contribution of the scalar field to the cosmological constant is dominant, by an order of magnitude, over all other fields throughout the Casimir regime and most of the Planckian regime. However as we consider larger and larger spatial sections of the Einstein universe

the temperature goes asymptotically to zero and the value of the cosmological constants for all the three fields approaching one another.

By presenting these results we do not claim that the cosmological constant problem is solved, but at least it may give us a glimpse of how the problem is trivially solved within the context of an Einstein static universe. Obviously the real universe is time-dependent, and a series of successive instantaneously static states of the dynamical universe lacks all the geometrodynamical effects; for example in this case particle production is one important effect that is lost.

V. Conclusions

The present study exhibited some features of the thermodynamical behavior of the Einstein static universe in presence of the neutrino field. In presenting the results of this investigation we stress the fact that due to the static nature of the Einstein universe, the following results are specific to the case considered and should not be taken to imply an evolving cosmological state. However, although not a realistic cosmological model, the Einstein universe provide a useful theoretical model to achieve better understanding of the interplay of spacetime curvature and of quantum field theoretic effects. The main findings of this work reinforces the results reported in I, with one additional point concerning the cosmological constant, these are

(1) The thermal development of the universe is a direct consequence of the state of its global curvature.

(2) The universe avoids the singularity at $T = 0$ through quantum effects (the Casimir effect) because of the non-zero value of $< T_{00} >_0$. A non-zero expectation value of the vacuum energy density always implies a symmetry breaking event.

(3) During the Casimir regime the universe is totally controlled by vacuum. The energy content of the universe is a function of its radius. Using the conformal relation between the static Einstein universe and the closed FRW universe [10], this result indicates that in a FRW model there would be a continuous creation of energy out of vacuum as long as the universe is expanding, a result which was confirmed by Parker long ago [21]. The steep, nearly vertical line in Fig. 2 suggests that the real universe started violently and had to relax later.

(4) Throughout the Casimir regime the cosmological constant Λ has a large value for all three fields, and is nearly constant, but once the universe goes to Planck regime the value of Λ drops sharply. In the realistic models for a dynamical universe, a large value for the cosmological constant during the very early stages of the universe is required so as to solve - via inflation- the horizon and flatness problem [19]. On the other hand, the value for the cosmological constant at present should be small in order to comply with observations. The results presented in this paper, though may not be directly related to a developing universe, provides us with the required behavior.

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Figure Caption

FIG 1. The temperature-radius relationship for the neutrino field in an Einstein universe.

FIG 2. Comparison beween the temperature-radius relationship for scalar, neutrino and photon fields. The dashed line is for the scalar field, the light solid line is for the neutrino field and the dark solid line is for the photon field.

FIG 3. The variation of the cosmological constant with the temperature of the universe resulting from the back reaction effect of the neutrino field.

FIG 4. Comparison between the contributions of the scalar, neutrino, and photon fields to the cosmological constant in an Einstein universe at finite temperatures. The dashed line is for the scalar field, the light solid line is for the neutrino field and the dark solid line is for the photon field.

T

Planck Regime

Casimir Regime

a

0   1   2   3   4

0.5   1   1.5   2